Lecture 27

Array Antennas, Fresnel Zone, Rayleigh Distance

We have seen that a simple Hertzian dipole has low directivity. The radiation pattern looks like that of a donut, and the directivity of the antenna is 1.5. Hence, for point-to-point communications, much power is wasted. However, the directivity of antennas can be improved if a group or array of dipoles can work cooperatively together. They can be made to constructively interfere in the desired direction, and destructively interfere in other directions to enhance their directivity. Since the far-field approximation of the radiation field can be made, and the relationship between the far field and the source is a Fourier transform relationship, clever engineering can be done borrowing knowledge from the signal processing area. After understanding the far-field physics, one can also understand many optical phenomena, such as how a laser pointer work. Many textbooks have been written about array antennas some of which are [131, 132].

27.1 Linear Array of Dipole Antennas

Antenna array can be designed so that the constructive and destructive interference in the far field can be used to steer the direction of radiation of the antenna, or the far-field radiation pattern of an antenna array. This is because the far field of a source is related to the source by a Fourier relationship. The relative phases of the array elements can be changed in time so that the beam of an array antenna can be steered in real time. This has important applications in, for example, air-traffic control. It is to be noted that if the current sources are impressed current sources, and they are the input to Maxwell's equations, and the fields are the output of the system, then we are dealing with a linear system here whereby linear system theory can be used. For instance, we can use Fourier transform to analyze the problem in the frequency domain. The time domain response can be obtained by inverse Fourier transform. This is provided that the current sources are impressed and they are not affected by the fields that they radiate.

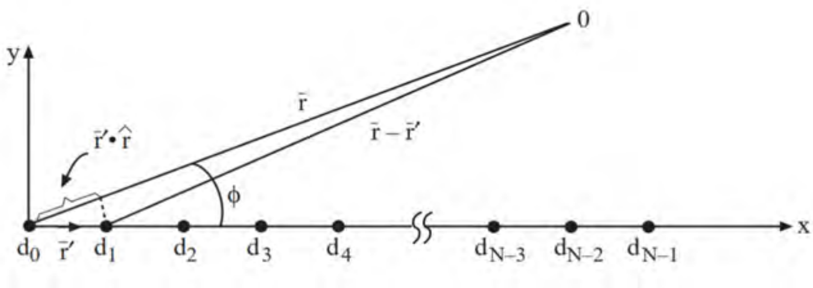

Figure 27.1: Schematics of a dipole array. To simplify the math, the far-field approximation can be used to find its far field or radiation field.

A simple linear dipole array is shown in Figure 27.1. First, without loss of generality, we assume that this is a linear array of point Hertzian dipoles aligned on the x axis. The current can then be described mathematically as follows:

$$\begin{aligned}\mathbf{J}(\mathbf{r}') = \hat{z}Il[&A_0\delta(x') + A_1\delta(x'-d_1) + A_2\delta(x'-d_2) + \cdots \\ &+ A_{N-1}\delta(x'-d_{N-1})]\delta(y')\delta(z')\end{aligned} \qquad (27.1.1)$$

The far field can be found using the approximate formula derived in the previous lecture, viz.,

$$\mathbf{A}(\mathbf{r}) \approx \frac{\mu e^{-j\beta r}}{4\pi r}\iiint_V d\mathbf{r}'\mathbf{J}(\mathbf{r}')e^{j\boldsymbol{\beta}\cdot\mathbf{r}'} \qquad (27.1.2)$$

27.1.1 Far-Field Approximation

The vector potential on the xy-plane in the far field, using the sifting property of delta function, yield the following equation for $\mathbf{A}(\mathbf{r})$ using (27.1.2),

$$\begin{aligned}\mathbf{A}(\mathbf{r}) &\cong \hat{z}\frac{\mu Il}{4\pi r}e^{-j\beta r}\iiint d\mathbf{r}'[A_0\delta(x') + A_1\delta(x'-d_1) + \cdots]\delta(y')\delta(z')e^{j\beta\mathbf{r}'\cdot\hat{r}} \\ &= \hat{z}\frac{\mu Il}{4\pi r}e^{-j\beta r}[A_0 + A_1 e^{j\beta d_1\cos\phi} + A_2 e^{j\beta d_2\cos\phi} + \cdots + A_{N-1}e^{j\beta d_{N-1}\cos\phi}]\end{aligned} \qquad (27.1.3)$$

In the above, for simplicity, we have assumed that the observation point is on the xy plane, or that $\mathbf{r} = \boldsymbol{\rho} = \hat{x}x + \hat{y}y$. Thus, $\hat{r} = \hat{x}\cos\phi + \hat{y}\sin\phi$. Also, since the sources are aligned on the x axis, then $\mathbf{r}' = \hat{x}x'$, and $\mathbf{r}'\cdot\hat{r} = x'\cos\phi$. Consequently, $e^{j\beta\mathbf{r}'\cdot\hat{r}} = e^{j\beta x'\cos\phi}$.

First, we let $d_n = nd$, implying an equally space array with distance d between adjacent elements. Then we let $A_n = e^{jn\psi}$, which assumes a progressively increasing phase shift between different elements. Such an antenna array is called a linear phase array. Thus, (27.1.3) in the above becomes

$$\mathbf{A}(\mathbf{r}) \cong \hat{z}\frac{\mu Il}{4\pi r}e^{-j\beta r}[1 + e^{j(\beta d\cos\phi+\psi)} + e^{j2(\beta d\cos\phi+\psi)} + \cdots \\ + e^{j(N-1)(\beta d\cos\phi+\psi)}] \tag{27.1.4}$$

With the simplifying assumptions, the above series can be summed in closed form.

27.1.2 Radiation Pattern of an Array

The above(27.1.4) can be summed in closed form using

$$\sum_{n=0}^{N-1} x^n = \frac{1-x^N}{1-x} \tag{27.1.5}$$

Then in the far field,

$$\mathbf{A}(\mathbf{r}) \cong \hat{z}\frac{\mu Il}{4\pi r}e^{-j\beta r}\frac{1-e^{jN(\beta d\cos\phi+\psi)}}{1-e^{j(\beta d\cos\phi+\psi)}} \tag{27.1.6}$$

Ordinarily, as shown previously, $\mathbf{E} \approx -j\omega(\hat{\theta}A_\theta + \hat{\phi}A_\phi)$. But since $\mathbf{A}$ is $\hat{z}$ directed, $A_\phi = 0$. Furthermore, on the xy plane, $E_\theta \approx -j\omega A_\theta = j\omega A_z$. As a consequence,

$$\begin{aligned} |E_\theta| &= |E_0|\left|\frac{1-e^{jN(\beta d\cos\phi+\psi)}}{1-e^{j(\beta d\cos\phi+\psi)}}\right|, \quad \mathbf{r}\to\infty \\ &= |E_0|\left|\frac{\sin\frac{N}{2}(\beta d\cos\phi+\psi)}{\sin\frac{1}{2}(\beta d\cos\phi+\psi)}\right|, \quad \mathbf{r}\to\infty \end{aligned} \tag{27.1.7}$$

The factor multiplying $|E_0|$ above is also called the array factor. The above can be used to plot the far-field pattern of an antenna array.

Equation (27.1.7) has an array factor that is of the form

$$\frac{|\sin Nx|}{|\sin x|}$$

This function appears in digital signal processing frequently, and is known as the digital sinc function [133]. The reason why this is so is because the far field is proportional to the Fourier transform of the current. The current in this case a finite array of Hertzian dipole, which is a product of a box function and infinite array of Hertzian dipole. The Fourier transform of such a current, as is well known in digital signal processing, is the digital sinc.

Plots of $|\sin 3x|$ and $|\sin x|$ are shown as an example and the resulting $\frac{|\sin 3x|}{|\sin x|}$ is also shown in Figure 27.2. The function peaks when both the numerator and the denominator of the digital sinc vanish. This happens when $x = n\pi$ for integer n.

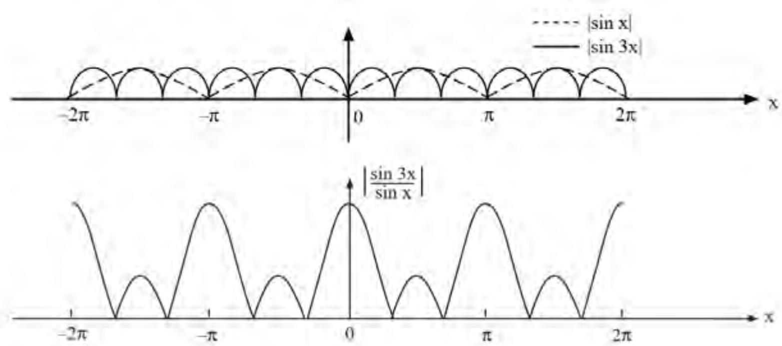

Figure 27.2: Plot of the digital sinc, $\frac{|\sin 3x|}{|\sin x|}$.

In equation (27.1.7), $x = \frac{1}{2}(\beta d \cos\phi + \psi)$. We notice that the **maximum** in (27.1.7) would occur if $x = n\pi$, or if

$$\beta d \cos\phi + \psi = 2n\pi, \qquad n = 0, \pm 1, \pm 2, \pm 3, \cdots \tag{27.1.8}$$

The **zeros** or **nulls** will occur at $Nx = n\pi$, or

$$\beta d \cos\phi + \psi = \frac{2n\pi}{N}, \qquad n = \pm 1, \pm 2, \pm 3, \cdots, \quad n \neq mN \tag{27.1.9}$$

For example,

Case I. $\psi = 0, \beta d = \pi$, principal maximum is at $\phi = \pm\frac{\pi}{2}$. If $N = 5$, nulls are at $\phi = \pm\cos^{-1}\left(\frac{2n}{5}\right)$, or $\phi = \pm 66.4^\circ, \pm 36.9^\circ, \pm 113.6^\circ, \pm 143.1^\circ$. The radiation pattern is seen to form lopes. Since $\psi = 0$, the radiated fields in the y direction are in phase and the peak of the radiation lope is in the y direction or the broadside direction. Hence, this is called a broadside array. The radiation pattern of such an array is shown in Figure 27.3.

Case II. $\psi = \pi, \beta d = \pi$, principal maximum is at $\phi = 0, \pi$. If $N = 4$, nulls are at $\phi = \pm\cos^{-1}\left(\frac{n}{2} - 1\right)$, or $\phi = \pm 120^\circ, \pm 90^\circ, \pm 60^\circ$. Since the sources are out of phase by 180°, and $N = 4$ is even, the radiation fields cancel each other in the broadside, but add in the x direction or the end-fire direction. Figure 27.4 shows the radiation pattern of such an array.

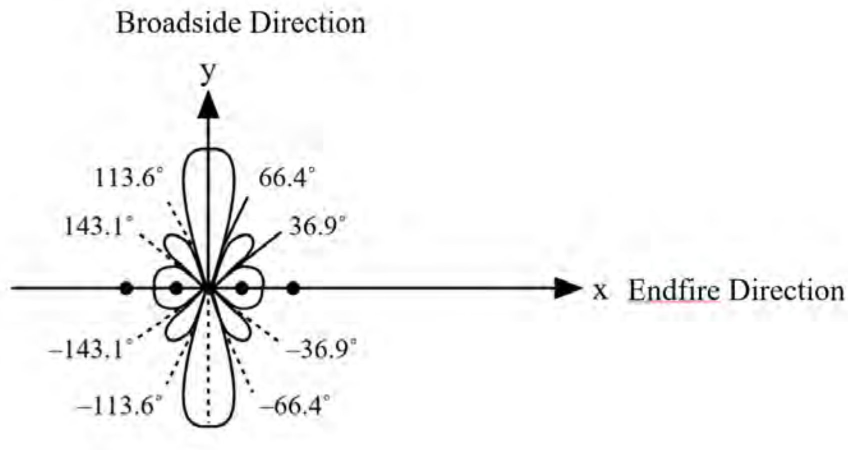

Figure 27.3: The radiation pattern of a five-element array. The broadside and endfire directions of the array are also labeled.

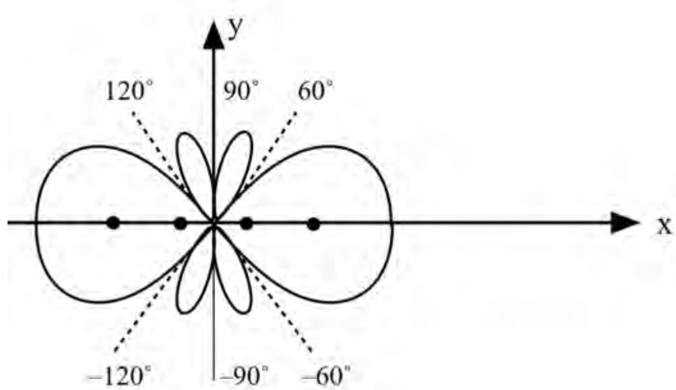

Figure 27.4: By changing the phase of the linear array, the radiation pattern of the antenna array can be changed.

From the above examples, it is seen that the interference effects between the different antenna elements of a linear array focus the power in a given direction. We can use linear array to increase the directivity of antennas. Moreover, it is shown that the radiation patterns can be changed by adjusting the spacings of the elements as well as the phase shift between them. The idea of antenna array design is to make the main lobe of the pattern to be much higher than the side lobes so that the radiated power of the antenna is directed along the main lobe or lobes rather than the side lobes. So side-lobe level suppression is an important goal of designing a highly directive antenna. Also, by changing the phase of the antenna elements in real time, the beam of the antenna can be steered in real time with no moving parts.

27.2 When is Far-Field Approximation Valid?

In making the far-field approximation in (27.1.3), it will be interesting to ponder when the far-field approximation is valid? That is, when we can approximate

$$e^{-j\beta|\mathbf{r}-\mathbf{r}'|} \approx e^{-j\beta r + j\beta \mathbf{r}'\cdot\hat{r}} \tag{27.2.1}$$

to arrive at (27.1.3). This is especially important because when we integrate over $\mathbf{r}'$, it can range over large values especially for a large array. In this case, $\mathbf{r}'$ can be as large as $(N-1)d$.

To answer this question, we need to study the approximation in (27.2.1) more carefully. First, we have

$$|\mathbf{r}-\mathbf{r}'|^2 = (\mathbf{r}-\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}') = r^2 - 2\mathbf{r}\cdot\mathbf{r}' + {r'}^2 \tag{27.2.2}$$

We can take the square root of the above to get

$$|\mathbf{r}-\mathbf{r}'| = r\left(1 - \frac{2\mathbf{r}\cdot\mathbf{r}'}{r^2} + \frac{{r'}^2}{r^2}\right)^{1/2} \tag{27.2.3}$$

Next, we use the Taylor series expansion to get, for small x, that

$$(1+x)^n \approx 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots \tag{27.2.4}$$

or that

$$(1+x)^{1/2} \approx 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots \tag{27.2.5}$$

We can apply this approximation by letting

$$x \doteq -\frac{2\mathbf{r}\cdot\mathbf{r}'}{r^2} + \frac{{r'}^2}{r^2}$$

To this end, we arrive at [1]

$$|\mathbf{r}-\mathbf{r}'| \approx r\left[1 - \frac{\mathbf{r}\cdot\mathbf{r}'}{r^2} + \frac{1}{2}\frac{{r'}^2}{r^2} - \frac{1}{2}\left(\frac{\mathbf{r}\cdot\mathbf{r}'}{r^2}\right)^2 + \cdots\right] \tag{27.2.6}$$

In the above, we have not kept every term of the x^2 terms by assuming that $r'^2 \ll \mathbf{r}'\cdot\mathbf{r}$, and terms much smaller than the last term in (27.2.6) can be neglected.

[1]The art of making such approximation is called perturbation expansion [40].

We can multiply out the right-hand side of the above to further arrive at

$$\begin{aligned}|\mathbf{r}-\mathbf{r}'| &\approx r - \frac{\mathbf{r}\cdot\mathbf{r}'}{r} + \frac{1}{2}\frac{{r'}^2}{r} - \frac{1}{2}\frac{(\mathbf{r}\cdot\mathbf{r}')^2}{r^3} + \cdots \\ &= r - \hat{r}\cdot\mathbf{r}' + \frac{1}{2}\frac{{r'}^2}{r} - \frac{1}{2r}(\hat{r}\cdot\mathbf{r}')^2 + \cdots \end{aligned} \tag{27.2.7}$$

The last two terms in the last line of (27.2.7) are of the same order.[2] Moreover, their sum is bounded by ${r'}^2/(2r)$ since $\hat{r}\cdot\mathbf{r}'$ is always less than r'. Hence, the far field approximation is valid if

$$\beta\frac{{r'}^2}{2r} \ll 1 \tag{27.2.8}$$

In the above, β is involved because the approximation has to be valid in the exponent, namely $\exp(-j\beta|\mathbf{r}-\mathbf{r}'|)$. If (27.2.8) is valid, then

$$e^{j\beta\frac{{r'}^2}{2r}} \approx 1$$

and then, the first two terms on the right-hand side of (27.2.7) suffice to approximate the left-hand side, which are the two terms we have kept in the far-field approximation.

27.2.1 Rayleigh Distance

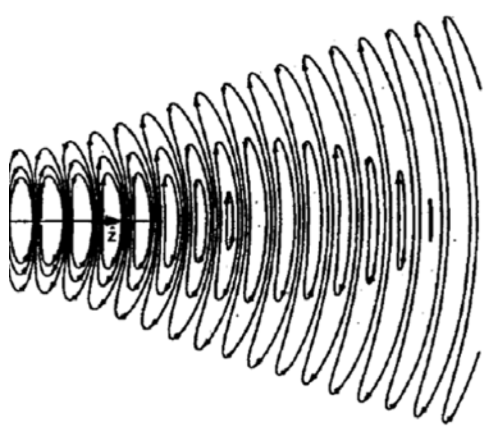

Figure 27.5: The right half of a Gaussian beam [75] displays the physics of the near field, the Fresnel zone, and the far zone. In the far zone, the field behaves like a spherical wave.

When a wave field leaves an aperture antenna, it can be approximately described by a Gaussian beam [75] (see Figure 27.5). Near to the antenna aperture, or in the near zone, it is

[2]The math parlance for saying that these two terms are approximately the same magnitude as each other.

approximately a plane wave with wave fronts parallel to the aperture surface. Far from the antenna aperture, or in the far zone, the field behaves like a spherical wave, with its typical wave front. In between is the Fresnel zone.

Consequently, after using that $\beta = 2\pi/\lambda$, for the far-field approximation to be valid, we need (27.2.8) to be valid, or that

$$r \gg \frac{\pi}{\lambda} r'^2 \tag{27.2.9}$$

If the aperture of the antenna is of radius W, then $r' < r_{\max} \cong W$ and the far field approximation is valid if

$$r \gg \frac{\pi}{\lambda} W^2 = r_R \tag{27.2.10}$$

If r is larger than this distance, then an antenna beam behaves like a spherical wave and starts to diverge. This distance r_R is also known as the Rayleigh distance. After this distance, the wave from a finite size source resembles a spherical wave which is diverging in all directions (see Figure 27.5). Also, notice that the shorter the wavelength λ, the larger is this distance. This also explains why a laser pointer works. A laser pointer light can be thought of radiation from a finite size source located at the aperture of the laser pointer as shall be shown using equivalence theorem later. The laser pointer beam remains collimated for quite a distance, before it becomes a divergent beam or a beam with a spherical wave front.

In some textbooks [31], it is common to define acceptable phase error to be $\pi/8$. The Rayleigh distance is the distance beyond which the phase error is below this value. When the phase error of $\pi/8$ is put on the right-hand side of (27.2.8), one gets

$$\beta \frac{r'^2}{2r} \approx \frac{\pi}{8} \tag{27.2.11}$$

Using the approximation, the Rayleigh distance is defined to be

$$r_R = \frac{2D^2}{\lambda} \tag{27.2.12}$$

where $D = 2W$ is the diameter of the antenna aperture. This concept is important in both optics and microwave.

27.2.2 Near Zone, Fresnel Zone, and Far Zone

Therefore, when a source radiates, the radiation field is divided into the near zone, the Fresnel zone, and the far zone (also known as the radiation zone, or the Fraunhofer zone in optics). The Rayleigh distance is the demarcation boundary between the Fresnel zone and the far zone. The larger the aperture of an antenna array is, the further one has to be in order to reach the far zone of an antenna. This distance becomes larger too when the wavelength is short. In the far zone, the far field behaves like a spherical wave, and its radiation pattern is proportional to the Fourier transform of the current.

In some sources, like the Hertzian dipole, in the near zone, much reactive energy is stored in the electric field or the magnetic field near to the source. This near zone receives reactive power from the source, which corresponds to instantaneous power that flows from the source, but is return to the source after one time harmonic cycle. Hence, a Hertzian dipole has input impedance that looks like that of a capacitor, because much of the near field of this dipole is in the electric field.

The field in the far zone carries power that radiates to infinity. As a result, the field in the near zone decays rapidly, but the field in the far zone decays as $1/r$ for energy conservation.